Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
`http://siba-ese.unisalento.it/index.php/ejasa/index`
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v10n1p29

Tolerance intervals and confidence intervals for the scale parameter of Pareto-Rayleigh distribution

By Godase, Shirke, Kashid

Published: 26 April 2017

Electronic Journal of Applied Statistical Analysis
Vol. 10, Issue 01, April 2017, 29-49
DOI: 10.1285/i20705948v10n1p29

Tolerance intervals and confidence intervals for the scale parameter of Pareto-Rayleigh distribution

Godase S.S.*[a], Shirke D.T.[b], and Kashid D.N.[b]

[a]*Department of Statistics, S. G. M. College, Karad, India*
[b]*Department of Statistics, Shivaji university, Kolhapur, India*

Published: 26 April 2017

In this paper we consider Pareto-Rayleigh distribution as an example of a Transformed-Transformer family of distributions defined by Alzaatreh et al. (2013b). We construct confidence intervals (CIs) and tolerance intervals (TIs) using generalized variable approach due to Weerahandi (1993) by using maximum likelihood estimator and modified maximum likelihood estimator of the scale parameter. Performance of both the intervals is studied using simulation and compared with the existing ones to exhibit superiority of the proposed intervals. Proposed confidence intervals and tolerance intervals are illustrated through real life data.

keywords: Transformed-transformer (T-X) family, Pareto-Rayleigh distribution, generalized pivotal quantity, confidence intervals and tolerance intervals.

1 Introduction

Pareto distribution has been widely used in modeling heavy-tailed distributions, such as income distribution. Many applications of the Pareto distribution in economics, biology and physics can be found in the literature. Schroeder et al. (2010) presented an application of the Pareto distribution in modeling disk drive sector errors. Mahmoudi (2011) discusses the beta generalized Pareto distribution with application to life time data. The Pareto distribution has been recognized as a suitable model for many non-negative socio-economic variables. Pareto distribution is useful in individual income,

*Corresponding author: suwarna_godase@rediffmail.com

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

family income and income before taxes etc. In literature various generalizations of the Pareto distribution have been derived such as Beta-Pareto distribution Akinsete et al. (2008).

Raqab and Kundu (2005) introduced the Rayleigh distribution in connection with a problem in the field of acoustics. An important characteristic of the Rayleigh distribution is that its hazard function is an increasing function of time. It means that when the failure times are distributed according to the Rayleigh law, an intense aging of the item takes place. Estimations, predictions and inferential issues for one parameter Rayleigh distribution have been extensively studied by several authors. Rayleigh distributions are useful in modeling and predicting tools in a wide variety of socio-economic contexts. The Rayleigh distribution has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. Potdar and Shirke (2013) have provided reliability estimation for the distribution of a k-unit parallel system with Rayleigh distribution as the component life distribution.

In many applied sciences such as medicine, engineering and finance amongst others, modeling and analyzing lifetime data are crucial. Several life time distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However there still remains many important problems, where the real data does not follow any of the classical or standard probability models. Pareto-Rayleigh is an example of Transformed-Transformer family (T-X family) of distributions, defined by Alzaatreh et al. (2013b). Also Alzaatreh et al. (2012) and Alzaatreh et al. (2013a) derived Gamma-Pareto distribution, Weibull-Pareto distribution and its applications.

In the present work, our focus is to provide confidence intervals and tolerance intervals based on maximum likelihood estimator (MLE) and modified maximum likelihood estimator (MMLE) of the parameter of Pareto-Rayleigh distribution. MLE in the present case is not available in the closed form and is to be obtained by using a suitable iterative method. Tiku (1967) obtained modified maximum likelihood (MML) equations which have explicit solutions by replacing the intractable terms by their linear approximations. Tiku and Suresh (1992) used the Taylor series expansion of the intractable terms in estimating the location and scale parameters in a symmetric family of distributions, which includes a number of well-known distributions such as normal, Students t etc. They also showed that the MML estimators, thus derived are asymptotically fully efficient for small samples. One may refer to Vaughan (1992), Suresh (1997) and Tiku (1967, 1968) for more details. In this article we use MLE and MML estimator to construct CIs and TIs.

A (β,1-γ) TI based on a sample is constructed so that it would include at least a proportion β of the sampled population with confidence 1-γ. Such a TI is usually referred to as β-content-(1-γ) coverage TI or simply (β, 1-γ) TI. A (β, 1-γ) upper tolerance limit (TL) is simply an (1-γ)th upper confidence limit for the (100γ)th percentile of the population and a (β, 1-γ) lower TL is an (1-γ)th lower confidence limit for the (100(1-γ))th percentile of the population. In this article, we are mainly concerned with one-sided TI using large sample (LS) approach and generalized variable (GV) approach for Pareto-Rayleigh distribution. Kumbhar and Shirke (2004) described TIs for lifetime distribution of k-unit parallel system, when component lifetime distribution is exponential. Liao et al. (2005) have proposed a method for constructing TIs in one-way random model based on the GV approach due to Weerahandi (1993).

Concept of GV has recently become popular in small sample inferences for complex problems such as Behrens-Fisher problem. These techniques have been shown to be efficient in specific distributions by using MLEs. The GV method was motivated by the fact that the small sample optimal CIs in statistical problems involving nuisance parameters may not be available. The method of generalized confidence interval (GCI) based on GV is used whenever standard pivotal quantities either do not exist or are difficult to obtain. Weerahandi (1993) introduced the concept of GCI. As described in the cited papers, GCI is based on the so-called generalized pivotal quantity (GPQ). For some problems, where the classical procedures are not optimal, GCI performs well. Krishnamoorthy and Mathew (2003) developed exact CI and tests for single lognormal mean using ideas of generalized p-values and GCIs. Guo and Krishnamoorthy (2005) explained a problem of interval estimation and testing for the difference between the quantiles of two populations using GV approach. Krishnamoorthy et al. (2006) explained generalized p-values and CIs with a novel approach for analyzing lognormal distributed exposure data. Krishnamoorthy et al. (2007) explained a problem of hypothesis testing and interval estimation of the reliability parameter in a stress-strength model involving two-parameter exponential distribution using GV approach. Verrill and Johnson (2007) considered confidence bounds and hypothesis tests for coefficient of variation of normal distribution. Kurian et al. (2008) have provided GCI for process capability indices in one-way random model. Krishnamoorthy and Lian (2012) derived generalized TIs for some general linear models based on GV approach. The literature survey reveals that during last ten years number of researchers have reported inference for the well known models using GV approach, which motivated us to consider the problem of generalized CI and generalized TI for Pareto-Rayleigh distribution. Rest of the paper is organized as follows.

In Section 2, the Pareto-Rayleigh distribution is considered and MLE and MMLE of the scale parameter are obtained. Section 3, provides CIs based on MLE and MMLE using LS procedure and GV approach. Section 4, provides TIs using LS procedure and GV approach. In section 5, the performance of the CIs and TIs using LS and GV approaches based on MLE and MMLE for small samples is investigated using simulations. Results of the simulation study have been reported in same section. In section 6, a real data set has been analyzed as an illustration.

2 Model and estimation of the scale parameter

Let F(.) be the cumulative distribution function (cdf) of any random variable X defined on $[0,\infty)$ and f (.) be the probability density function (pdf) of a random variable

T, defined on [0,∞). The cdf of the T-X family of distributions defined by Alzaatreh et al. (2013b) is given by

$$G(x) = \int_0^{-log(1-F(x))} f(t)dt \tag{1}$$

Alzaatreh et al. (2013b) named this family of distributions the Transformed-Transformer family (or T-X family) of distributions. If a random variable T follows the Pareto distribution type IV with parameter α then pdf of T is given by,

$$f(t) = \alpha(1+t)^{-(\alpha+1)} \qquad t > 0, \alpha > 1 \tag{2}$$

If a random variable X follows the Rayleigh distribution with parameter σ then cdf of X is given by,

$$F(x) = 1 - \exp(-x^2/2\sigma^2) \qquad \sigma > 0, x > 0 \tag{3}$$

Using (1), (2) and (3), the cdf of Pareto-Rayleigh distribution (as a member of T-X family) is given by,

$$G(x) = \int_0^{x^2/2\sigma^2} \alpha(1+t)^{(-\alpha+1)}dt = 1 - \left(1 + \frac{x^2}{2\sigma^2}\right)^{-\alpha} \qquad x > 0, \alpha > 1, \sigma > 0 \tag{4}$$

The pdf of Pareto-Rayleigh distribution is given by,

$$g(x) = \frac{\alpha}{\sigma^2}x\left(1 + \frac{x^2}{2\sigma^2}\right)^{(-\alpha-1)} \qquad x > 0, \alpha > 1, \sigma > 0, \tag{5}$$

where α is the known shape parameter and σ is the unknown scale parameter. In this article, we are mainly concerned with CIs and TIs of Pareto-Rayleigh distribution using MLE and MMLE of the scale parameter σ .

2.1 Maximum Likelihood Estimation

The pdf of the Pareto-Rayleigh distribution with scale parameter σ and shape parameter α is given by (5).
Let $X_1, X_2, ..., X_n$ be a random sample of size n obtained from Pareto-Rayleigh distribution. By taking the derivative of log likelihood equation, the MLE of the scale parameter σ is the solution of the following equation.

$$\frac{\partial lnL}{\partial\sigma} = 0 = -2n + \frac{\alpha+1}{\sigma^2}\sum_{i=1}^{n}\frac{x_i^2}{(1 + \frac{x_i^2}{2\sigma^2})} = 0.$$

This equation shows that maximum likelihood estimator of $\sigma(\hat{\sigma_n})$ is an iterative solution which can be obtained by suitable iterative method like bisection method. Then Fisher information about σ is given by

$$I = -nE\left[\frac{\partial^2 lnf(x,\alpha,\sigma)}{\partial\sigma^2}\right] = \frac{2n(3\alpha+2)}{\sigma^2(\alpha+2)} - \frac{2n}{\sigma^2}$$

2.2 Modified Maximum Likelihood Estimation

We have seen that MLE of scale parameter σ is not in the closed form as the likelihood equation is intractable. To overcome this difficulty, we use MML method of estimation (Tiku and Suresh (1992)) to find the estimate of scale parameter σ. This can be done by first expressing the maximum likelihood equation in terms of order statistics and then replacing the intractable terms by their linear approximation.
Maximum likelihood equation can be written as

$$\frac{\partial lnL}{\partial \sigma} = 0 = -2n + (\alpha+1)\sum_{i=1}^{n} \frac{z_i^2}{1+\frac{z_i^2}{2}} = 0 \tag{6}$$

where

$$z_i = \frac{x_i}{\sigma}.$$

The maximum likelihood equation (6) does not have explicit solution for scale parameter σ. This is due to the fact that the term

$$g(z_i) = \frac{z_i^2}{1+\frac{z_i^2}{2}}$$

is intractable. To formulate MML equation, which has explicit solution, we express this equation in terms of order statistics that is

$$\frac{\partial lnL}{\partial \sigma} = 0 = -2n + (\alpha+1)\sum_{i=1}^{n} \frac{z_{(i)}^2}{1+\frac{z_{(i)}^2}{2}} = 0 \tag{7}$$

where $z_{(i)}$ is the order statistic of the sample observations x_i,(i=1,2,...,n). The second step is to linearize equation (7) by using Taylor series expansion around the quantile point of G. The linearization is done in such a way that the derived MMLE retains all the desirable asymptotic properties of the MLEs. Thus we have,

$$g(z_{(i)}) = \frac{z_{(i)}^2}{1+\frac{z_{(i)}^2}{2}} = a_i + b_i z_{(i)} \tag{8}$$

The third step is to obtain the modified maximum likelihood equation by incorporating (8) in (7), that is

$$\frac{\partial lnL}{\partial \sigma} = 0 = \frac{\partial lnL^*}{\partial \sigma} = -2n + (\alpha+1)\sum (a_i + b_i z_{(i)}) \tag{9}$$

The solution to equation (9) is the MMLE, which is given by

$$\hat{\sigma} = \frac{\sum b_i x_{(i)}}{\frac{n}{\alpha+1} - \sum a_i} \tag{10}$$

where

$$b_i = g^{'}(z_{(i)}), a_i = g(z_{(i)}) - b_i z_{(i)}$$

One may refer to Tiku and Suresh (1992) and Suresh (2004) for more details.
In the following, we shall see two methods of finding confidence intervals for scale parameter σ using MLE and MMLE.
Lemma 2.1: Distribution of $(\frac{\hat{\sigma}_n}{\sigma})$ and $(\frac{\hat{\sigma}}{\sigma})$, both are free from σ where $\hat{\sigma}_n$ is MLE and $\hat{\sigma}$ is MMLE of σ.
Proof: The proof is similar to the one given by Gulati and Mi (2006). This lemma can be used to find GPQ.

3 Confidence Intervals

3.1 Large sample confidence interval

Theorem: As $n \to \infty$,

$$\sqrt{n}(\hat{\sigma} - \sigma) \longrightarrow N_2(0, I^{-1})$$

where I is the Fisher information given in section (2.1).
Proof: Proof follows from asymptotic properties of MLEs under regularity conditions. Since σ is unknown, I is estimated by replcing σ by its MLE or MMLE and this can be used to obtain the asymptotic CI of σ.
The approximate $100(1-\tau)\%$ asymptotic confidence interval (ACI) for σ is given by

$$\left(\hat{\sigma} \pm z_{1-\tau/2}\sqrt{\frac{I^{-1}}{n}}\right) \tag{11}$$

where $z_{1-\tau/2}$ is the $(1-\tau/2)^{th}$ quantile of the standard normal distribution.

According to Tiku and Suresh (1992) the derived MMLEs retain all the desirable asymptotic properties of the MLEs. Hence simply by replacing MLEs with MMLEs we can obtain confidence interval using large sample approach based on MMLE. We denote this interval by I_1.

3.2 Generalized variable approach

The concept of a generalized confidence interval is due to Weerahandi (1993). One may also refer to Weerahandi (2013) for a detailed discussion along with numerous examples. Consider a random variable X (scalar or vector) whose distribution $g(x, \sigma, \delta)$ depends on a scalar parameter of interest σ and a nuisance parameter (parameter that is not of direct inferential interest)δ, where δ could be a vector. Suppose we are interested in computing a confidence interval for scale parameter σ. Let, x denotes the observed value of X. To construct a GCI for σ, we first define a GPQ, $T(X; x, \sigma, \delta)$ which is a function of random variable X, its observed data x, the parameters σ and δ. A quantity $T(X; x, \sigma, \delta)$ is required to satisfy the following two conditions.
i) For a fixed x, the probability distribution of $T(X; x, \sigma, \delta)$ is free of unknown parameters σ and δ;
ii) The observed value of $T(X; x, \sigma, \delta)$, namely $T(x; x, \sigma, \delta)$ is simply σ.

The percentiles of $T(X; x, \sigma, \delta)$ can then be used to obtain confidence intervals for σ. Such confidence intervals are referred to as generalized confidence intervals. For example, if $T_{1-\tau}$ denotes the $100_{1-\tau}$ th percentile of $T(X; x, \sigma, \delta)$, then $T_{1-\tau}$ is a generalized upper confidence limit for σ. Therefore $100(1-\tau)\%$ two-sided GCI for parameter σ is given by

$$(T_{\tau/2}, T_{1-\tau/2}).$$

Define GPQ as

$$T_1(X; x, \sigma) = \frac{\hat{\sigma}_o}{\frac{\hat{\sigma}}{\sigma}},$$

where $\hat{\sigma}_o$ is the MLE obtained using observed data. We note the following:
i) Distribution of $T_1(X; x, \sigma)$ is free from σ, which follows from Lemma (2.1) and
ii) $T_1(X; x, \sigma) = \sigma$, since for observed data, $\hat{\sigma} = \hat{\sigma}_o$. A GCI based on $T_1(X; x, \sigma)$ is obtained by using the following algorithm. The GCI is denoted by I_2.

I. Algorithm to obtain GCI for σ using GPQ

1. Input n, N, α, σ, τ.
2. Generate independently and identically distributed observations $(U_1, U_2, ..., U_n)$ from U(0,1).
3. For the given value of the parameter σ , set

$$x_i = \sqrt{2\sigma^2((1-U_i)^{-1/\alpha} - 1)} \qquad for \quad i = 1, 2, ..., n.$$

Then $(x_1, x_2, ..., x_n)$ is random sample of size n from Pareto-Rayleigh distribution with parameter σ.
4. Based on observations in step 3, obtain MLE of σ (say $\hat{\sigma}_o$), using bisection method.
5. Generate random sample of size n from Pareto-Rayleigh distribution with parameter σ=1.
6. Based on observations in step 5, obtain MLE of σ (say $\hat{\sigma}$)using bisection method.
7. Compute GPQ, $T_1 = \frac{\hat{\sigma}_o}{\hat{\sigma}}$
8. Repeat steps (5) to (7) N times, so as to get $T_{11}, T_{12}, ..., T_{1N}$.
9. Arrange T_{1i}^s in an ascending order. Denote them by $T_{(11)}, T_{(12)}, ..., T_{(1N)}$.
10. Compute a $100(1-\tau)\%$ GCI for σ as $(T_{(1,([(\tau_2)N])}, T_{(1,([(1-\tau_2)N]))})$.

Extending above algorithm one can estimate coverage probability of the proposed GCI. In the above algorithm, we can replace MLE by MMLE and obtain GCI based on MMLE.

4 Tolerance Intervals

4.1 Large Sample Tolerance Intervals

There are two types of tolerance intervals namely β-expectation tolerance interval (TI) and β-content-(1-γ) coverage tolerance interval.

4.1.1 β-expectation TI for the distribution function G (.; σ)

Let $X_\beta(\sigma)$ be the lower quantile of order β of the distribution function G (.; σ). Then, we have

$$X_\beta(\sigma) = \sqrt{2\sigma^2\{(1-\beta)^{-1/\alpha} - 1\}}$$

Since σ is unknown, we replace it by its MLE. Hence maximum likelihood estimate of $X_\beta(\sigma)$ is given by

$$X_\beta(\hat{\sigma}) = \sqrt{2\hat{\sigma}^2\{(1-\beta)^{-1/\alpha} - 1\}} \tag{12}$$

having an approximate upper β-expectation TI for G (.; σ) as

$$J_1(X) = (0, X_\beta(\hat{\sigma})) \tag{13}$$

We approximate $E[G(X_\beta(\sigma);\sigma)]$ using Atwood (1984) and is given as

$$E[G(X_\beta(\hat{\sigma});\sigma)] \approx \beta - 0.5F_{02}Var(\hat{\sigma}) + \frac{F_{01}Var(\hat{\sigma})F_{11}}{F_{10}} \tag{14}$$

where $F_{10} = \frac{\partial G(x;\sigma)}{\partial x}$, $F_{01} = \frac{\partial G(x;\sigma)}{\partial \sigma}$, $F_{11} = \frac{\partial^2 G(x;\sigma)}{\partial x \partial \sigma}$, $F_{02} = \frac{\partial^2 G(x;\sigma)}{\partial \sigma^2}$ with $x = X_\beta(\sigma)$ and all the derivatives are evaluated at X_β and σ. We can replace MLE by MMLE and obtain β-expectation TI for G (.; σ) based on MMLE. Simulated and approximate values of expected coverage of $J_1(X)$ using MLE and MMLE have been reported in section 5 for different values of n, β and α.

4.1.2 β-content-(1-γ) coverage Tolerance Interval

Let $J_2(X) = (0, D\hat{\sigma})$ be an upper β-content-(1-γ) coverage TI for the distribution having distribution function (4). The constant $D(>0)$ for $\beta\epsilon(0,1)$, $\gamma\epsilon(0,1)$ is to be determined such that

$$P\{G(D\hat{\sigma};\sigma) \leq \beta\} = 1-\gamma$$

That is

$$P\left\{\hat{\sigma} \leq \sigma\frac{\sqrt{2\{(1-\beta)^{-1/\alpha} - 1\}}}{D}\right\} = 1-\gamma \tag{15}$$

Using asymptotic normality of $\hat{\sigma}$ equation (15) can be equivalently written as

$$P\left\{Z \leq (\frac{\sigma}{var(\sigma)})\frac{\sqrt{2\{(1-\beta)^{-1/\alpha} - 1\}}}{D} - 1\right\} = 1-\gamma,$$

where Z follows N(0,1). This gives

$$D = \frac{\sqrt{2\{(1-\beta)^{-1/\alpha}-1\}}}{1+\frac{var(\sigma)}{\sigma}z_{1-\gamma}}$$

Hence, an upper tolerance limit of β-content-(1-γ) coverage tolerance interval ($J_2(X)$) is given by

$$U(X) = \hat{\sigma}\left\{\frac{\sqrt{2\{(1-\beta)^{-1/\alpha}-1\}}}{1+\frac{var(\sigma)}{\sigma}z_{1-\gamma}}\right\} \tag{16}$$

4.2 Generalized Tolerance Intervals

The problem of computing a one-sided tolerance limit reduces to that of computing a one-sided confidence limit for the percentile of the relevant probability distribution. That is a $(\beta,(1-\gamma))$ upper tolerance limit is simply an (1-γ)th upper confidence limit for the (100β)th percentile of the population. It is easily seen that a $(\beta,(1-\gamma))$ upper tolerance limit for G (.; σ) is simply a 100(1-γ)% upper confidence limit for $\sqrt{2\sigma^2[(1-\beta)^{-1/\alpha}-1]}$. We use the GV approach for obtaining the aforementioned upper confidence limit.
Let $\hat{\sigma}_o$ is the MLE obtained using observed data. The GPQ for constructing a confidence interval for σ is given by $T_1(X;x,\sigma) = \frac{\hat{\sigma}_o}{\hat{\sigma}_i/\sigma}$,i=1,2,...,N. The GPQ for $\sqrt{2\sigma^2[(1-\beta)^{-1/\alpha}-1]}$ is given by

$$T_2 = \frac{\hat{\sigma}_o}{\hat{\sigma}_i/\sigma}\sqrt{2[(1-\beta)^{-1/\alpha}-1]}, \qquad i = 1,2,...,N.$$

The $(1-\gamma)$th quantile of T_2 is a $(1-\gamma)$th generalized upper confidence bound for $\sqrt{2\sigma^2[(1-\beta)^{-1/\alpha}-1]}$. Hence $(\beta,(1-\gamma))$ upper tolerance limit for $G(.;\sigma)$ is $(0, T_{2,1-\gamma})$. A generalized tolerance interval based on $T_2(X;x,\sigma)$is obtained by using the following algorithm.

II. Algorithm to obtain Generalized Tolerance Interval for $G(.;\sigma)$ using GPQ

1. Input n, N, $\alpha,\sigma,\beta,\gamma$.
2. Input random sample of size n from Pareto-Rayleigh distribution with an unknown parameter σ.
3. Based on observations in step 2, obtain MLE of σ (say $\hat{\sigma}_o$), using bisection method.
4. Generate random sample of size n from Pareto-Rayleigh distribution with parameter $\sigma = 1$.
5. Based on observations in step 4, obtain MLE of σ (say $\hat{\sigma}$), using bisection method.
6. Compute GPQ,
$T_2 = \frac{\hat{\sigma}_o}{\hat{\sigma}_i/\sigma}\sqrt{2[(1-\beta)^{-1/\alpha}-1]}, \qquad i = 1,2,...,N.$
7. Repeat steps (4) to (6) N times, so as to get $T_{21}, T_{22}, ..., T_{2N}$.
8. Arrange $T_{2i}'s$ in an ascending order. Denote them by $T_{21}, T_{22}, ..., T_{2N}$
9. Compute an upper tolerance limit of generalized TI $J_2(X) = (0, T_{2,1-\gamma})$.

Extending above algorithm one can estimate coverage probability of the proposed generalized TI. In the above algorithm, we can replace MLE by MMLE and obtain generalized TI, based on MMLE.

5 Numerical and simulation study

We conduct extensive simulation experiments to evaluate performance of CIs (LS approach and GV approach) based on MLE and MMLE. We choose different values of σ, β, n and α. Results are tabulated in Tables 1-2. Figures in the 1st row are based on MLE, while figures in the 2nd row are based on MMLE. From Tables 1-2, we observe that simulated coverage of GCI does not differ significantly whether it can be computed from MLE as well as MMLE. However, large sample approach underestimates the coverage probabilities for most of the scenarios, especially when the sample size is small and (or) the parameter σ is large. Also the performance of the proposed GCI does not depend on σ. As the sample size is large, the two estimators (MLE, MMLE) are equally efficient.

We investigate coverage (numerical and simulation) of β-expectation TI for Pareto-Rayleigh distribution with $\alpha = 3$ and β= 0.90, 0.95,0.99 by using MLE and MMLE. Figures in the 1st row are based on MLE, while figures in the 2nd row are based on MMLE. An upper β-expectation tolerance limit is given in equation (12). Results of the simulation study for the β-expectation tolerance interval, which is tabulated in Table 3, indicate that, the estimated expectation and simulation mean for small sample size are marginally lower than the nominal value. As the sample size increases, the performance of tolerance intervals improves. We observe the following from Table 3.

The estimated expectation of the coverage of the approximate β-expectation tolerance intervals shows satisfactory result for large n. Estimated expectation and simulated mean of the coverage increase as sample size n increase. Estimated expectation and simulated mean of the coverage remains same as shape parameter increases. Simulated mean of the coverage for small sample size is below nominal level.

A simulation study of an upper β-content- (1- γ) coverage TI, having an upper limit (16) is also conducted, for σ=1, 2 and for known values of n, β, α and γ. In this simulation study 5000 samples from G (.; σ) were generated and for each of the samples U(X) was computed, for different combinations of β, σ , γ. The proportion of samples for which $\sqrt{2\sigma^2[(1-\beta)^{-1/\alpha}-1]}$ exceeded U(X) was computed 100 times and the mean of these 100 proportions is taken as simulated value of γ. The simulation study for the generalized TI was carried out as algorithm (II). Tables 5-6 give the simulated values of confidence level γ when σ=1, 2 respectively. The proposed confidence interval performs satisfactory for small to moderate sample sizes. These intervals are superior to the asymptotic confidence intervals.

Table 1: Mean coverage of Confidence Intervals (using MLE and MMLE) for transformed transformer (Pareto-Rayleigh) distribution I_1) Large Sample procedure I_2) Generalized variable approach when σ=1.0, α=2.0

coverage	0.90		0.95		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2
2	0.8604	0.9012	0.8962	0.9445	0.931	0.9887
	0.8652	0.9004	0.8932	0.9434	0.9291	0.9894
3	0.8723	0.9024	0.8931	0.9552	0.9458	0.9947
	0.8651	0.8994	0.9162	0.9558	0.9454	0.9990
4	0.8741	0.9025	0.9041	0.9537	0.9548	0.9889
	0.8735	0.9036	0.9217	0.9534	0.9634	0.9910
5	0.8811	0.9028	0.9147	0.9502	0.9615	0.9963
	0.8879	0.9047	0.9181	0.9532	0.9664	0.9924
6	0.8805	0.9022	0.9251	0.9534	0.9538	0.9917
	0.8898	0.9019	0.9352	0.9564	0.9644	0.9934
7	0.8841	0.9047	0.9284	0.9521	0.9665	0.9937
	0.8897	0.9024	0.9294	0.9588	0.9724	0.9918
8	0.8889	0.9068	0.9281	0.9588	0.9735	0.9919
	0.8962	0.9088	0.9462	0.9531	0.9654	0.9934
9	0.8771	0.9021	0.9354	0.9529	0.9814	0.9935
	0.8981	0.9011	0.9381	0.9574	0.9684	0.9928
10	0.8910	0.9024	0.9474	0.9534	0.9715	0.9915
	0.8907	0.9024	0.9474	0.9538	0.9764	0.9966
15	0.8888	0.9008	0.9364	0.9536	0.9775	0.9919
	0.8946	0.9064	0.9464	0.9587	0.9814	0.9921
30	0.8947	0.9055	0.9484	0.9537	0.9865	0.9926
	0.9014	0.9027	0.9562	0.9564	0.984	0.9987
50	0.8932	0.9064	0.9314	0.9528	0.9845	0.9928
	0.9016	0.9033	0.9414	0.9508	0.9894	0.9980

Table 2: Mean coverage of Confidence Intervals (using MLE and MMLE) for transformed transformer (Pareto-Rayleigh) distribution I_1) Large Sample procedure I_2) Generalized variable approach when σ=2.0, α=2.0

coverage	0.90		0.95		0.99	
n	I_1	I_2	I_1	I_2	I_1	I_2
2	0.8605	0.8992	0.8894	0.9487	0.9312	0.9887
	0.8625	0.8988	0.8905	0.9425	0.9219	0.9805
3	0.8736	0.8989	0.9008	0.9432	0.9448	0.9928
	0.8715	0.9080	0.9020	0.9485	0.9321	0.9865
4	0.8781	0.9030	0.9172	0.9506	0.9504	0.9889
	0.8724	0.9053	0.9251	0.9538	0.9603	0.9932
5	0.8921	0.9021	0.9204	0.9524	0.9614	0.9962
	0.8829	0.9026	0.9148	0.9519	0.9668	0.9937
6	0.8938	0.9062	0.9224	0.9522	0.9534	0.9932
	0.8905	0.9028	0.9321	0.9537	0.9617	0.9919
7	0.8908	0.9081	0.9318	0.9540	0.9624	0.9984
	0.8842	0.9024	0.9304	0.9531	0.9724	0.9941
8	0.8921	0.9061	0.9326	0.9565	0.9735	0.9958
	0.8955	0.9008	0.9428	0.9528	0.9625	0.9935
9	0.8881	0.9073	0.9306	0.9535	0.9814	0.9931
	0.8918	0.9026	0.9325	0.9522	0.9757	0.9984
10	0.8962	0.9083	0.9341	0.9557	0.9795	0.9922
	0.8925	0.9034	0.9487	0.9565	0.9743	0.9957
15	0.8994	0.9043	0.9412	0.9548	0.9724	0.9943
	0.8997	0.9050	0.9427	0.9566	0.9817	0.9980
30	0.8934	0.9018	0.9474	0.9541	0.9887	0.9957
	0.8906	0.9024	0.9438	0.9564	0.9814	0.9972
50	0.8956	0.9028	0.9518	0.9561	0.9822	0.9964
	0.8941	0.9084	0.958	0.9534	0.9878	0.9955

Table 3: Simulated mean and estimated expectation of the coverage of approximate β-expectation TI using MLE and MMLE for transformed transformer (Pareto-Rayleigh) distribution.

	$\alpha = 3$							
	$\beta(\sigma = 1.0)$				$\beta(\sigma=2.0)$			
n	0.90	0.95	0.97	0.99	0.90	0.95	0.97	0.99
2	0.8112	0.8595	0.9065	0.9515	0.8315	0.8712	0.9172	0.9521
	(0.8251)	(0.8459)	(0.8902)	(0.9625)	(0.8451)	(0.8652)	(0.9251)	(0.9534)
	0.0.7921	0.8888	0.9127	0.9318	0.8298	0.8585	0.9275	0.9434
	(0.7912)	(0.8892)	(0.9021)	(0.9425)	(0.8329)	(0.8625)	(0.9265)	(0.9547)
3	0.8568	0.8996	0.9384	0.9592	0.8436	0.9118	0.9418	0.9637
	(0.8495)	(0.8825)	(0.9365)	(0.9469)	(0.8492)	(0.9028)	(0.9356)	(0.9645)
	0.8465	0.9124	0.9386	0.9544	0.8494	0.8917	0.9374	0.9568
	(0.8520)	(0.9062)	(0.9255)	(0.9528)	(0.8574)	(0.9054)	(0.9487)	(0.9534)
4	0.8716	0.9142	0.9499	0.9756	0.8333	0.9014	0.9375	0.9725
	(0.8724)	(0.9028)	(0.9589)	(0.9728)	(0.8365)	(0.9124)	(0.9425)	(0.9824)
	0.0.8588	0.8923	0.9491	0.9693	0.8514	0.9151	0.9438	0.9695
	(0.8459)	(0.8902)	(0.9425)	(0.9714)	(0.8495)	(0.9024)	(0.9457)	(0.9748)
5	0.8632	0.9151	0.9454	0.9737	0.8697	0.9222	0.9537	0.9786
	(0.8794)	(0.9215)	(0.9316)	(0.9722)	(0.8724)	(0.9365)	(0.9633)	(0.9748)
	0.0.8610	0.9244	0.9558	0.9611	0.8712	0.9023	0.9449	0.9659
	(0.8705)	(0.9145)	(0.9420)	(0.9784)	(0.8790)	(0.9124)	(0.9584)	(0.9721)
6	0.8754	0.9359	0.9565	0.9859	0.8725	0.9179	0.9539	0.9791
	(0.8715)	(0.9302)	(0.9536)	(0.9850)	(0.8837)	(0.9274)	(0.9521)	(0.9701)
	0.0.8665	0.9197	0.9523	0.9750	0.8774	0.9178	0.9494	0.9814
	(0.8714)	(0.9028)	(0.9577)	(0.9815)	(0.8791)	(0.9154)	(0.9524)	(0.9825)
7	0.8668	0.9417	0.9647	0.9847	0.8839	0.9346	0.9689	0.9817
	(0.8628)	(0.9459)	(0.9619)	(0.9824)	(0.8829)	(0.9435)	(0.9752)	(0.9932)
	0.0.8577	0.9278	0.9569	0.9794	0.8746	0.9244	0.9516	0.9735
	(0.8459)	(0.9160)	(0.9654)	(0.9728)	(0.8859)	(0.9284)	(0.9654)	(0.9849)
8	0.8889	0.9296	0.9692	0.9859	0.8654	0.9328	0.9614	0.9872
	(0.8749)	(0.9239)	(0.9628)	(0.9822)	(0.8735)	(0.9475)	(0.9672)	(0.9824)
	0.0.8945	0.9344	0.9674	0.9765	0.8747	0.9325	0.9577	0.9840
	(0.8891)	(0.9385)	(0.9587)	(0.9711)	(0.8815)	(0.9425)	(0.9657)	(0.9864)

Table 4: Simulated mean and estimated expectation of the coverage of approximate β-expectation TI using MLE and MMLE for transformed transformer (Pareto-Rayleigh) distribution. Continued

	$\alpha = 3$							
	$\beta(\sigma = 1.0)$				$\beta(\sigma$=2.0)			
n	0.90	0.95	0.97	0.99	0.90	0.95	0.97	0.99
9	0.8787	0.9277	0.9616	0.9872	0.8781	0.9342	0.9645	0.9896
	(0.8892)	(0.9258)	(0.9621)	(0.9826)	(0.8724)	(0.9451)	(0.9754)	(0.9833)
	0.0.8914	0.9389	0.9647	0.9813	0.8790	0.9294	0.9592	0.98140
	(0.8928)	(0.9225)	(0.9618)	(0.9837)	(0.8739)	(0.9321)	(0.9625)	(0.9802)
10	0.8856	0.9265	0.9588	0.9885	0.8838	0.9416	0.9671	0.9829
	(0.8821)	(0.9368)	(0.9548)	(0.9814)	(0.8902)	(0.9478)	(0.9784)	(0.9820)
	0.0.8831	0.9314	0.9692	0.9848	0.8765	0.9333	0.9664	0.9817
	(0.8834)	(0.9425)	(0.9664)	(0.9834)	(0.8834)	(0.9401)	(0.9725)	(0.9849)
15	0.8919	0.9346	0.9631	0.9914	0.8769	0.9314	0.9657	0.9885
	(0.8940)	(0.9365)	(0.9748)	(0.9889)	(0.8729)	(0.9365)	(0.9781)	(0.9804)
	0.0.8994	0.9475	0.9715	0.9886	0.8836	0.9379	0.9698	0.9851
	(0.8921)	(0.9428)	(0.9708)	(0.9948)	(0.8924)	(0.9425)	(0.9748)	(0.9834)
30	0.8837	0.9428	0.9779	0.9927	0.8993	0.9517	0.9685	0.9952
	(0.9024)	(0.9458)	(0.9645)	(0.9917)	(0.9028)	(0.9538)	(0.9677)	(0.9889)
	0.0.9016	0.9492	0.9737	0.9879	0.8865	0.9495	0.9769	0.9826
	(0.9099)	(0.9359)	(0.9721)	(0.9950)	(0.8949)	(0.9584)	(0.9780)	(0.9887)
50	0.9028	0.9492	0.9695	0.9987	0.9014	0.9532	0.9746	0.9949
	(0.9082)	(0.9584)	(0.9635)	(0.9980)	(0.9147)	(0.9502)	(0.9722)	(0.9924)
	0.0.9092	0.9514	0.9753	0.9914	0.8916	0.9534	0.9753	0.99140
	(0.9158)	(0.9506)	(0.9748)	(0.9940)	(0.9025)	(0.9524)	(0.9824)	(0.9914)

Table 5: Coverage probabilities of Tolerance Intervals for Pareto-Rayleigh distribution I_1) Large sample procedure I_2) Generalized variable approach σ=1.0, α=2.0

	γ=0.90				γ=0.95			
coverage	β=0.90		β=0.95		β=0.90		β=0.95	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
2	0.6672	0.9021	0.6432	0.8924	0.5549	0.9448	0.5521	0.9449
	0.6451	0.9028	0.6544	0.8920	0.5441	0.9459	0.5549	0.9428
3	0.7984	0.8992	0.7971	0.8935	0.7231	0.9432	0.7461	0.9452
	0.7846	0.9021	0.7869	0.8922	0.7266	0.9458	0.7361	0.9488
4	0.8156	0.9034	0.8194	0.9031	0.8224	0.9538	0.8564	0.9468
	0.8356	0.9038	0.8347	0.8977	0.8319	0.9458	0.8479	0.9585
5	0.8448	0.9049	0.8435	0.9028	0.8815	0.9562	0.8714	0.9562
	0.8544	0.9028	0.8539	0.9024	0.8819	0.9564	0.8854	0.9534
6	0.8639	0.9125	0.8556	0.9034	0.8901	0.9537	0.9032	0.9538
	0.8634	0.9037	0.8619	0.9021	0.9034	0.9538	0.9035	0.9566
7	0.8644	0.9028	0.8598	0.9055	0.8974	0.9539	0.9074	0.9533
	0.8664	0.8997	0.8686	0.9029	0.9096	0.9532	0.9083	0.9580
8	0.8492	0.9064	0.8429	0.9064	0.9087	0.9654	0.9097	0.9582
	0.8706	0.9035	0.8695	0.9068	0.9144	0.9538	0.9157	0.9534
9	0.8493	0.9038	0.8239	0.9031	0.9124	0.9587	0.9015	0.9524
	0.8716	0.9028	0.8714	0.9024	0.9188	0.9458	0.9183	0.9531
10	0.8614	0.9034	0.8497	0.9124	0.9235	0.9482	0.9032	0.9654
	0.8744	0.9046	0.8724	0.8992	0.9203	0.533	0.9240	0.9587
15	0.8718	0.9029	0.8544	0.8997	0.9114	0.9588	0.9225	0.9528
	0.8798	0.9029	0.8792	0.9034	0.9272	0.9526	0.9284	0.9575
30	0.8790	0.9184	0.8831	0.9024	0.9278	0.9537	0.9315	0.9521
	0.8890	0.9088	0.8872	0.9098	0.9352	0.9538	0.9361	0.9648
50	0.9031	0.9028	0.8951	0.9089	0.9445	0.9526	0.9294	0.9588
	0.8924	0.9090	0.8905	0.9044	0.9449	0.9524	0.9482	0.9584

Table 6: Coverage probabilities of Tolerance Intervals for Pareto-Rayleigh distribution I_1) Large sample procedure I_2) Generalized variable approach σ=2.0, α=2.0

	γ=0.90				γ=0.95			
coverage	β=0.90		β=0.95		β=0.90		β=0.95	
n	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
2	0.6431	0.8925	0.6831	0.8902	0.5621	0.9485	0.5331	0.9487
	0.6401	0.8959	0.6598	0.8988	0.5741	0.9458	0.5521	0.9415
3	0.7811	0.8933	0.8032	0.8953	0.7378	0.9428	0.7394	0.9458
	0.7822	0.8954	0.7852	0.8954	0.7451	0.9462	0.7421	0.9402
4	0.8180	0.9024	0.8394	0.8934	0.8584	0.9521	0.8441	0.9567
	0.8370	0.9024	0.8350	0.9028	0.8566	0.9532	0.8504	0.9435
5	0.8334	0.9028	0.8532	0.9028	0.8893	0.9439	0.9012	0.9548
	0.8537	0.9058	0.8569	0.8937	0.8920	0.9531	0.9135	0.9520
6	0.8521	0.9024	0.8421	0.9054	0.9132	0.9511	0.9035	0.9448
	0.8629	0.9034	0.8694	0.9024	0.9230	0.9489	0.9127	0.9537
7	0.8592	0.9022	0.8584	0.9027	0.8894	0.9560	0.9136	0.9580
	0.8645	0.9031	0.8651	0.9037	0.8904	0.9518	0.9198	0.9582
8	0.8754	0.9065	0.8725	0.9013	0.9052	0.9538	0.9158	0.9502
	0.8779	0.9157	0.8633	0.9026	0.9124	0.9588	0.9230	0.9531
9	0.8531	0.9021	0.8649	0.9157	0.9012	0.9528	0.8869	0.9531
	0.8732	0.9055	0.8724	0.9027	0.9124	0.9575	0.8920	0.9565
10	0.8421	0.9128	0.8564	0.9024	0.8954	0.9582	0.9117	0.9521
	0.8724	0.9071	0.8734	0.9147	0.9280	0.9548	0.9228	0.9533
15	0.8621	0.9034	0.8697	0.8948	0.9235	0.9489	0.9174	0.9587
	0.8799	0.9028	0.8788	0.9088	0.9284	0.9521	0.9257	0.9502
30	0.8674	0.9089	0.8587	0.9028	0.9151	0.9568	0.9239	0.9654
	0.8854	0.9021	0.8876	0.9056	0.9329	0.9588	0.9360	0.9536
50	0.8981	0.9080	0.8879	0.9027	0.9294	0.9586	0.9487	0.9537
	0.8952	0.9072	0.8991	0.9076	0.9428	0.9548	0.9510	0.9531

6 Real life Data Analysis

In this section we present a data analysis of the strength data reported by Bader and Priest (1982). It is already observed by Durham and Padgett (1997) that Weibull model does not work well in this case. Surles and Padgett (1998), Surles and Padgett (2001) and Raqab and Kundu (2005) observed that generalized Rayleigh works quite well for this strength data. Also Raqab and Kundu (2005) observed goodness of fit of the three-parameter generalized exponential distribution to this data set based on modified MLEs.

For illustrative purpose we also consider the same transformed data set as considered by Raqab and Kundu (2005), the single fibers of 10 mm in gauge length with sample size 63. Data set is presented below:
0.101,0.332,0.403,0.428,0.457,0.550,0.561,0.596,0.597,0.645,0.654,0.674,0.718,0.722, 0.725,0.732,0.775,0.814,0.816,0.818,0.824,0.859,0.875,0.938,0.940,1.056,1.117,1.128, 1.137,1.137,1.177,1.196,1.230,1.325,1.339,1.345,1.420,1.423,1.435,1.443,1.464,1.472, 1.494,1.532,1.546,1.577,1.608,1.635,1.693,1.701,1.737,1.754,1.762,1.828,2.052,2.071, 2.086,2.171,2.224,2.227,2.425,2.295,3.220.

First we would like to compute the MLEs of the unknown parameters. The MLE of σ is obtained as 2.036426 and the MLE of α becomes 5.036467 with the associated log-likelihood value as -57.67675. We plot the empirical survival function and the fitted survival function. We used the Kolmogorov-Smirnov (K-S) test for this data set. K-S distance between the fitted Pareto-Rayleigh and empirical cumulative distribution function is 0.094377 and the associated p-value is 0.8431. Therefore, it indicates that the Pareto-Rayleigh model provides reasonable fit to this data set.
Based on the estimates of α and σ, the confidence intervals (using LS and GV approach) are given in the Table 7.

Table 7: Confidence intervals (using LS and GV approach) for strength data.

Coverage	Using Estimator	Using LS approach(ACI)	Using GV approach(GCI)
90%	MLE	(1.787754,2.285098) Length=0.4973437	(1.914382,2.197766) Length=0.283384
	MMLE	(1.786463,2.283543) Length=0.4970807	(1.402121,1.805491) Length=0.4033698
95%	MLE	(1.711940,2.360913) Length=0.6489728	(1.893224,2.246205) Length=0.3496253
	MMLE	(1.737967,2.332039) Length=0.594072	(1.366712,1.836193) Length=0.4694808
99%	MLE	(1.645223,2.427629) Length=0.7824065	(1.852753,2.313607) Length=0.4608534
	MMLE	(1.644007,2.425999) Length=0.7819927	(1.884905,2.698532) Length=0.713627

Therefore, in this case it is clear that the GV approach provides confidence interval having shortest length than the LS approach.
We also evaluated (0.90, 0.90) and (0.95, 0.95) upper tolerance limits for this data set using LS and GV approach. They are 2.921123 (2.875510) and 3.694097(3.56269) respectively. Bracketed tolerance limit is using GV approach.

Table 8: The maximum likelihood estimates and Kolmogorov-Smirnov statistics and p-values for strength data.

The model	MLEs of the parameters	Log-likelihood	K-S statistic	p-value
Generalized Rayleigh	$\hat{\beta}$=1.4216,$\hat{\lambda}$=0.8598	-50.22	0.12	0.2845
Three parameter GE	$\hat{\beta}$=4.3586,$\hat{\lambda}$=1.8303,$\hat{\alpha}$=6.5469	-110.01	0.0933	0.643
Pareto- Rayleigh	$\hat{\alpha}$=5.036467,$\hat{\sigma}$=2.036426	-57.67675	0.094377	0.8431

It is clear from the Table 8 that based on the K-S statistic, the proposed Pareto-Rayleigh model provides a better fit than generalized Rayleigh and three parameter generalized Exponential models to this specific data set. Although, it is not guaranteed that the proposed model always provides a better fit than the other models.

7 Conclusions

In this paper we have considered interval estimation (confidence interval and tolerance interval) using maximum likelihood estimator and modified maximum likelihood estimator in Pareto-Rayleigh distribution (Transformed-Transformer family) based on generalized variable approach. We have compared these generalized intervals with asymptotic intervals. The proposed confidence intervals perform satisfactory for small to moderate sample sizes. These intervals are superior to the asymptotic intervals. The performance of the interval estimation using modified maximum likelihood estimators are also quite satisfactory. One real data analysis has been performed and it is observed that the proposed model provides a better fit than some of the existing models.

8 Acknowledgement

We thank referee for the valuable comments and suggestions. The second and third authors would like to acknowledge the support of University Grants Commission, New

Delhi under Special Assistance Programme to carry out the research work.

References

Akinsete, A., Famoye, F., and Lee, C. (2008). The beta-pareto distribution. *Statistics*, 42(6):547–563.

Alzaatreh, A., Famoye, F., and Lee, C. (2012). Gamma-pareto distribution and its applications. *Journal of Modern Applied Statistical Methods*, 11(1):7.

Alzaatreh, A., Famoye, F., and Lee, C. (2013a). Weibull-pareto distribution and its applications. *Communications in Statistics-Theory and Methods*, 42(9):1673–1691.

Alzaatreh, A., Lee, C., and Famoye, F. (2013b). A new method for generating families of continuous distributions. *Metron*, 71(1):63–79.

Atwood, C. L. (1984). Approximate tolerance intervals, based on maximum likelihood estimates. *Journal of the American Statistical Association*, 79(386):459–465.

Bader, M. and Priest, A. (1982). Statistical aspects of fibre and bundle strength in hybrid composites. *Progress in science and engineering of composites*, pages 1129–1136.

Durham, S. and Padgett, W. (1997). Cumulative damage models for system failure with application to carbon fibers and composites. *Technometrics*, 39(1):34–44.

Gulati, S. and Mi, J. (2006). Testing for scale families using total variation distance. *Journal of Statistical Computation and Simulation*, 76(9):773–792.

Guo, H. and Krishnamoorthy, K. (2005). Comparison between two quantiles: The normal and exponential cases. *Communications in StatisticsSimulation and Computation®*, 34(2):243–252.

Krishnamoorthy, K. and Lian, X. (2012). Closed-form approximate tolerance intervals for some general linear models and comparison studies. *Journal of Statistical Computation and Simulation*, 82(4):547–563.

Krishnamoorthy, K. and Mathew, T. (2003). Inferences on the means of lognormal distributions using generalized p-values and generalized confidence intervals. *Journal of statistical planning and inference*, 115(1):103–121.

Krishnamoorthy, K., Mathew, T., and Ramachandran, G. (2006). Generalized p-values and confidence intervals: A novel approach for analyzing lognormally distributed exposure data. *Journal of occupational and environmental hygiene*, 3(11):642–650.

Krishnamoorthy, K., Mukherjee, S., and Guo, H. (2007). Inference on reliability in two-parameter exponential stress–strength model. *Metrika*, 65(3):261–273.

Kumbhar, R. and Shirke, D. (2004). Tolerance limits for lifetime distribution of k-unit parallel system. *Journal of Statistical Computation and Simulation*, 74(3):201–213.

Kurian, K., Mathew, T., and Sebastian, G. (2008). Generalized confidence intervals for process capability indices in the one-way random model. *Metrika*, 67(1):83–92.

Liao, C., Lin, T., and Iyer, H. (2005). One-and two-sided tolerance intervals for general balanced mixed models and unbalanced one-way random models. *Technometrics*, 47(3):323–335.

Mahmoudi, E. (2011). The beta generalized pareto distribution with application to lifetime data. *Mathematics and computers in Simulation*, 81(11):2414–2430.

Potdar, K. and Shirke, D. (2013). Reliability estimation for the distribution of a k-unit parallel system with rayleigh distribution as the component life distribution. In *International Journal of Engineering Research and Technology*, volume 2. ESRSA Publications.

Raqab, M. Z. and Kundu, D. (2005). Comparison of different estimators of p [y¡ x] for a scaled burr type x distribution. *Communications in StatisticsSimulation and Computation®*, 34(2):465–483.

Schroeder, B., Damouras, S., and Gill, P. (2010). Understanding latent sector errors and how to protect against them. *ACM Transactions on storage (TOS)*, 6(3):9.

Suresh, R. (1997). On approximate likelihood estimators in censored normal samples. *Gujarat Statistical Review*, 24:21–28.

Suresh, R. (2004). Estimation of location and scale parameters in a two-parameter exponential distribution from a censored sample.

Surles, J. and Padgett, W. (1998). Inference for p (y¡ x) in the burr type x model. *Journal of Applied Statistical Science*, 7(4):225–238.

Surles, J. and Padgett, W. (2001). Inference for reliability and stress-strength for a scaled burr type x distribution. *Lifetime Data Analysis*, 7(2):187–200.

Tiku, M. (1967). Estimating the mean and standard deviation from a censored normal sample. *Biometrika*, 54(1-2):155–165.

Tiku, M. (1968). Estimating the parameters of normal and logistic distributions from censored samples. *Australian & New Zealand Journal of Statistics*, 10(2):64–74.

Tiku, M. and Suresh, R. (1992). A new method of estimation for location and scale parameters. *Journal of Statistical Planning and Inference*, 30(2):281–292.

Vaughan, D. C. (1992). On the tiku-suresh method of estimation. *Communications in Statistics-theory and Methods*, 21(2):451–469.

Verrill, S. and Johnson, R. A. (2007). Confidence bounds and hypothesis tests for normal distribution coefficients of variation. *Communications in StatisticsTheory and Methods*, 36(12):2187–2206.

Weerahandi, S. (1993). Generalized confidence intervals. *Journal of the American Statistical Association*, 88(423):899–905.

Weerahandi, S. (2013). *Exact statistical methods for data analysis.* Springer Science & Business Media.